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Corrector estimates in homogenization of a nonlinear transmission problem for diffusion equations in connected domains

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This paper is devoted to the homogenization of a nonlinear transmission problem stated in a two-phase domain. We consider a system of linear diffusion equations defined in a periodic domain consisting of two disjoint phases that are both connected sets separated by a thin interface. Depending on the field variables, at the interface, nonlinear conditions are imposed to describe interface reactions. In the variational setting of the problem, we prove the homogenization theorem and a bidomain averaged model. The periodic unfolding technique is used to obtain the residual error estimate with a first-order corrector.

KEYWORDS
bidomain model, corrector estimates, diffusion problem, nonlinear transmission conditions, periodic unfolding technique

MSC CLASSIFICATION
35B27; 35M10; 82C24

1 | INTRODUCTION

We consider coupled linear parabolic equations describing the diffusion of two species in two different phases of one physical domain separated by a thin periodic interface. The coupling of the species arises via nonlinear transmission conditions at the interface, which model surface reactions. Nonlinear interface reactions are relevant, for instance, in electrochemistry, see, eg, Landstorfer et al[1] for adsorption and solvation effects at metal-electrolyte interfaces, and Efendiev et al[2] for electro-chemical reactions in lithium-ion batteries.

The characteristic length scale of the periodic cell is given by the homogenization parameter $\varepsilon > 0$. The main objective is to derive a macroscopic model for vanishing ε, where both phases are connected sets. The limit bidomain model is given via two coupled parabolic equations defined in the macroscopic domain describing the diffusion of the two species in each phase and reactions at the interface. In the case of connected-connected domains, we exploit the existence of a continuous extension operator from the periodic domain to the whole domain following.[3,4]

A qualitative homogenization result for reaction-diffusion systems with nonlinear transmission conditions has recently been obtained in Gahn et al.[5] The limit in the microscopic equations is derived rigorously in the sense of the two-scale convergence, however, without corrector estimates. There also exists a vast literature on transmission problems with linear interface conditions, eg, Donato et al[6] and Donato and Monsurro.[7] See references therein for the case of elliptic equations as well as the extensions of the homogenization result to parabolic equations in Jose[8] and to nonlinear monotone transmission conditions in Donato and Le Nguyen.[9] For the treatment of oscillating third boundary conditions, we refer to Belyaev et al[10] and Oleinik and Shaposhnikova.[11]

Within elecktrokinetic modeling (see Allaire et al[12]), in previous studies,[13-16] there were considered generalized Poisson-Nernst-Planck (PNP) models over two-phase domains accounting for interface reactions. The corresponding PDE system obeys a structure of the gradient flow; see, eg, other works.[17-19] The paper[20] considers the homogenization over a two-phase domain for static PNP equations and homogeneous interface conditions. In Kovtunenko and Zubkova,[21] residual error estimates for the averaged monodomain solution with first-order correctors were justified under the simplifying assumption that the flux across the interface is of order $O(\varepsilon^2)$.

In this paper, however, we are mainly interested in quantitative asymptotic results supported by corrector estimates. There exist many articles on the derivation of error estimates for different classes of reaction-diffusion systems, eg, other works,[22-25] exploiting a higher regularity of the limit solution and the continuous extension operator from a perforated domain. Moreover, unfolding-based error estimates have been proven for linear, elliptic transmission problems in Reichelt,[26] for reaction-diffusion systems with linear boundary conditions in perforated domains in Muntean and Reichelt,[27] and for systems with nonlinear interface conditions in a two-phase domain in Fatima et al.[28] The latter results are based on the quantification of the periodicity defect for the periodic unfolding operator in Griso,[29,30] and they hold without assuming higher regularity for the corrector problem.

Our approach uses the periodic unfolding method introduced in Cioranescu et sl[31] and further refined in Franců[32] and Mielke and Timofte.[33] To make our error estimates rigorous, we have to assume higher regularity for the limit solutions as well as for the correctors solving the local cell problems. This additional regularity for the limit problem is in line with established homogenization results by, eg, literature.[34-36] Our result provides residual error estimates with a first-order corrector of order $\sqrt{\varepsilon}$, which is (generally) optimal for H^1-estimates up to an Lipschitz boundary, whereas in Fatima et al,[28] the error is of order $\varepsilon^{1/4}$. For this task, we apply the Poincaré inequality in periodic domains (see Lemma 2) and the uniform extension in connected periodic domains (see Lemma 3).

The paper is structured as follows: In Section 2, we formulate the transmission problem and all relevant assumptions. In Section 3, we prove the existence of solutions to our model and provide a priori estimates. In Sections 4 and 5, we define the periodic unfolding operator and provide important properties as well as first asymptotic results. In Section 6, we state and prove our main result on the residual error estimates.

2 | SETTING OF THE TRANSMISSION PROBLEM

For a fixed homogenization parameter $\varepsilon > 0$, we consider a macroscopic domain Ω consisting of two subsets Ω_1^ε, Ω_2^ε, which are disjoint by a thin interface Γ^ε. The both components Ω_i^ε are assumed to be connected such that $|\partial\Omega_i^\varepsilon \cap \partial\Omega| \neq 0$. By $|\partial\Omega_i^\varepsilon \cap \partial\Omega|$, we mean the surface measure of points where the boundaries of Ω_i^ε and Ω will meet.

We make the following geometric assumptions.

(D1) The reference domain $\Omega \subset \mathbb{R}^d$ is a d-dimensional hyperrectangle, $d \geqslant 2$, ie, it is

$$\Omega = \prod_{k=1}^{d}(a_k, b_k), \quad a_k < b_k \quad \text{and} \quad a_k, b_k \in \mathbb{R}.$$

This assumption suffices to split Ω into periodic cells in (D3).

(D2) The *unit cell* $Y = (0,1)^d$ consists of two open, connected subsets Y_1 and Y_2, which have Lipschitz continuous boundaries ∂Y_1, ∂Y_2 and are disjoint by the interface $\Gamma = \partial Y_1 \cap \partial Y_2$. We assume the reflection symmetry, ie,

$$\partial Y_i \cap \{y_k = 0\} = \partial Y_i \cap \{y_k = 1\}$$

for $k = 1, \dots, d$, $i = 1, 2$. This assumption allows us to define periodic functions on Y_i in (29). Let n_1 and n_2 denote the unit normal vectors at the respective boundaries ∂Y_1 and ∂Y_2. Every normal is chosen outward from the domain, and it does not depend on scaling by ε.

(D3) For $\varepsilon > 0$, we introduce the decomposition of a point $x \in \mathbb{R}^d$ as

$$x = \varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon \left\{ \frac{x}{\varepsilon} \right\} \tag{1}$$

into the floor part $\left\lfloor \frac{x}{\varepsilon} \right\rfloor \in \mathbb{Z}^d$ and the fractional part $\left\{ \frac{x}{\varepsilon} \right\} \in Y$. According to (1), let the set of integer vectors

$$I_\varepsilon = \{\lambda \in \mathbb{Z}^d \mid \varepsilon(\lambda + y) \in \Omega \quad \text{for all} \quad y \in Y\}$$

denote the numbering of local cells inside Ω. We call ε an admissible parameter, if the reference domain Ω from (D1) can be partitioned periodically into the local cells as follows:

$$\overline{\Omega} = \bigcup_{\lambda \in I_\varepsilon} \varepsilon(\lambda + \overline{Y}). \tag{2}$$

For a treatment of small boundary layers, see Reichelt.[37, lemma 2.3.3]

(D4) As a consequence of (D1) to (D3), the periodic components Ω_1^ε and Ω_2^ε and their interface Γ^ε are determined via

$$\overline{\Omega_i^\varepsilon} = \bigcup_{\lambda \in I_\varepsilon} \overline{Y_i^\lambda}, \quad Y_i^\lambda = \varepsilon(\lambda + Y_i), \quad \Gamma^\varepsilon = \partial\Omega_1^\varepsilon \cap \partial\Omega_2^\varepsilon. \tag{3}$$

By this, the outward normal vectors n_i^ε at $\partial\Omega_i^\varepsilon$ coincide with the normal vectors n_i at ∂Y_i for $i = 1, 2$ and do not depend on the scaling ε. The interface Γ^ε is a Lipschitz continuous manifold.

For admissible $\varepsilon > 0$, time $t \in (0, T)$ with the final time $T > 0$ fixed, the space variable $x \in \overline{\Omega_1^\varepsilon} \bigcup \overline{\Omega_2^\varepsilon}$ in the two-component domain, we consider a nonlinear transmission problem for $u_i^\varepsilon(t, x)$, $i = 1, 2$, such that

$$\partial_t u_i^\varepsilon - \operatorname{div}(A_i^\varepsilon \nabla u_i^\varepsilon) = 0 \quad \text{in} \quad \Omega_i^\varepsilon, \tag{4a}$$

$$A_i^\varepsilon \nabla u_i^\varepsilon \cdot n_i = \varepsilon g_i(u_1^\varepsilon, u_2^\varepsilon) \quad \text{on} \quad \Gamma^\varepsilon, \tag{4b}$$

$$u_i^\varepsilon = 0 \quad \text{on} \quad \partial\Omega_i^\varepsilon \cap \partial\Omega, \tag{4c}$$

$$u_i^\varepsilon = u_i^{\text{in}} \quad \text{as} \quad t = 0. \tag{4d}$$

The notation ∂_t stands for the time derivative, ∇ for the spatial gradient, and "$\cdot$" for the scalar product in $\mathbb{R}^d$. Below, we explain in detail the terms entering the system (4). We note that $|\Gamma^\varepsilon| = O(1/\varepsilon)$; therefore, the scaling ε in (4b) appears naturally just compensating the longer interface.

(A1) The diffusivity matrices $A_i(y) \in L^\infty(Y_i; \mathbb{R}^{d\times d}_{sym})$, $i = 1, 2$, are symmetric, uniformly bounded and elliptic: There exist $0 < \alpha \leqslant \beta$ such that

$$\alpha|\xi|^2 \leqslant A_i(y)\xi \cdot \xi \leqslant \beta|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^d, \quad \text{a.e.} \quad y \in Y_i. \tag{5}$$

The matrices entering (4a) to (4c) are defined as $A_i^\varepsilon(x) = A_i\left(\left\{\frac{x}{\varepsilon}\right\}\right)$ according to the notation (1) and are assumed to be periodic.

In the transmission conditions (4b), the functions $g_i : \mathbb{R}^2 \mapsto \mathbb{R}$, $i = 1, 2$, describe interface reactions and are assumed to satisfy

(G1) the uniform growth condition: there exists $K_g > 0$ such that

$$|g_i(u_1, u_2)| \leqslant K_g, \quad \text{for all} \quad u_1, u_2 \in \mathbb{R}; \tag{6}$$

(G2) the Lipschitz continuity: There exists $L_g \geqslant 0$ such that

$$|g_i(u_1, u_2) - g_i(v_1, v_2)| \leqslant L_g\left(|u_1 - v_1| + |u_2 - v_2|\right), \tag{7}$$

for all $u_i, v_i \in \mathbb{R}$, $i = 1, 2$.

The linear diffusion equations (4a) are supported by the standard, homogeneous Dirichlet boundary conditions (4c) and the initial data (4d) for given $u_i^{\text{in}} \in L^2(\Omega)$, $i = 1, 2$.

We introduce the variational formulation of the problem (4) as follows: find $u_i^\varepsilon \in \mathcal{U}_i^\varepsilon$, $i = 1, 2$, in the search (solution) space

$$\mathcal{U}_i^\varepsilon = \{u \in C(0,T;L^2(\Omega_i^\varepsilon)) \cap L^2(0,T;H^1(\Omega_i^\varepsilon)) : \quad \partial_t u \in L^2(0,T;H^1(\Omega_i^\varepsilon)^*),\ u = 0 \quad \text{on} \quad \partial\Omega_i^\varepsilon \cap \partial\Omega\},$$

satisfying the initial condition (4d) and the nonlinear equation

$$\int_0^T \left(\langle \partial_t u_i^\varepsilon, v_i \rangle_{\Omega_i^\varepsilon} + \int_{\Omega_i^\varepsilon} A_i^\varepsilon \nabla u_i^\varepsilon \cdot \nabla v_i \, dx \right) dt = \int_0^T \int_{\Gamma^\varepsilon} \varepsilon g_i(u_1^\varepsilon, u_2^\varepsilon) v_i \, d\sigma_x \, dt, \tag{8}$$

for all test functions v_i from the test space

$$\mathcal{V}_i^\varepsilon := \{v \in L^2(0,T;H^1(\Omega_i^\varepsilon)),\ v = 0 \quad \text{on} \quad \partial\Omega_i^\varepsilon \cap \partial\Omega\}.$$

The notation $H^1(\Omega_i^\varepsilon)^*$ in $\mathcal{U}_i^\varepsilon$ stands for the topologically dual space to $H^1(\Omega_i^\varepsilon)$, and $\langle\cdot,\cdot\rangle_{\Omega_i^\varepsilon}$ denotes the duality between them.

3 | WELL-POSEDNESS

This section provides the existence of weak solutions in the sense of variational formulation for the microscopic problem (8).

Theorem 1 (Well-posedness).

(i) The unique solution $u_i^\varepsilon \in \mathcal{U}_i^\varepsilon$ to the nonlinear transmission problem (8) exists and satisfies the following a priori estimate:

$$\begin{aligned} \|u_i^\varepsilon\|_{\mathcal{U}_i^\varepsilon}^2 &:= \|u_i^\varepsilon\|_{C(0,T;L^2(\Omega_i^\varepsilon))}^2 + \|u_i^\varepsilon\|_{L^2(0,T;H^1(\Omega_i^\varepsilon))}^2 + \|\partial_t u_i^\varepsilon\|_{L^2(0,T;H^1(\Omega_i^\varepsilon)^*)}^2 \\ &\leqslant C_1 \|u_i^{in}\|_{L^2(\Omega_i^\varepsilon)}^2 + C_2 K_g^2 + C_3, \quad C_1, C_2, C_3 \geqslant 0, \end{aligned} \tag{9}$$

uniformly in $\varepsilon \in (0, \varepsilon_0)$ for $\varepsilon_0 > 0$ sufficiently small.

(ii) Under assumptions on positivity of the initial data $u_i^{in} > 0$ everywhere in $\overline{\Omega}$, the solution u_i^ε is positive at least locally in time, and $u_i^\varepsilon \geqslant 0$ at any time under the assumption of the positive production rate from RoubÍček[38]:

$$g_i(u_1^\varepsilon, u_2^\varepsilon)(u_i^\varepsilon)^- = 0, \tag{10}$$

where $(u_i^\varepsilon)^- = -\min(0, u_i^\varepsilon)$ stands for the negative part of the function.

Proof.

(i) To prove existence of the solution, we apply the Tikhonov-Schauder fixed point theorem. We iterate (8) starting with the suitable initialization $u_i^{m_0} = u_i^{\text{in}}$, $m_0 \in \mathbb{N}$, $i = 1, 2$.
For $m > m_0$, $m \in \mathbb{N}$, a solution $u_i^m \in \mathcal{U}_i^\varepsilon$ can be found, which satisfies the initial data (4d) and the linearized equations

$$\int_0^T \left(\langle \partial_t u_i^m, v_i \rangle_{\Omega_i^\varepsilon} + \int_{\Omega_i^\varepsilon} A_i^\varepsilon \nabla u_i^m \cdot \nabla v_i \, dx \right) dt = \int_0^T \int_{\Gamma^\varepsilon} \varepsilon g_i^{m-1} v_i \, d\sigma_x \, dt, \tag{11}$$

for all test functions $v_i \in \mathcal{V}_i^\varepsilon$, using the notation $g_i^{m-1} := g_i(u_1^{m-1}, u_2^{m-1})$ for short. We can test (11) with $v_i = u_i^m$ leading to

$$\int_0^T \left(\langle \partial_t u_i^m, u_i^m \rangle_{\Omega_i^\varepsilon} + \int_{\Omega_i^\varepsilon} A_i^\varepsilon \nabla u_i^m \cdot \nabla u_i^m \, dx \right) dt = \int_0^T \int_{\Gamma^\varepsilon} \varepsilon g_i^{m-1} u_i^m \, d\sigma_x \, dt. \tag{12}$$

We estimate the integral in the right-hand side of (12) applying weighted Young inequality with a weight $\frac{2\delta}{K_{\rm tr}} > 0$, the trace theorem (25) below, and the growth condition (6):

$$\left|\int_0^T \int_{\Gamma^\varepsilon} \varepsilon g_i^{m-1} u_i^m \, d\sigma_x \, dt\right| \leqslant \frac{\delta\varepsilon}{K_{\rm tr}} \int_0^T \int_{\Gamma^\varepsilon} (u_i^m)^2 \, d\sigma_x \, dt + \frac{\varepsilon K_{\rm tr}}{4\delta} \int_0^T \int_{\Gamma^\varepsilon} (g_i^{m-1})^2 \, d\sigma_x \, dt \leqslant \delta \|u_i^m\|^2_{L^2(0,T;H^1(\Omega_i^\varepsilon))} + C, \quad (13)$$

where $C = \frac{K_{\rm tr}}{4\delta} K_{\rm g}^2 T \varepsilon |\Gamma^\varepsilon| = O(1)$ with a constant $K_{\rm tr}$ from the trace theorem (25) and $K_{\rm g}$ from (6). Expressing the first term in the left-hand side of (12) by the chain rule as $\langle \partial_t u_i^m, u_i^m \rangle = \frac{1}{2}\frac{d}{dt}\|u_i^m\|_{L^2(\Omega_i^\varepsilon)}$, using the uniform ellipticity (5) of A_i^ε and the estimate (13), this follows

$$\frac{1}{2}\frac{d}{dt} \int_0^T \int_{\Omega_i^\varepsilon} (u_i^m)^2 \, dx \, dt + (\alpha - \delta) \|\nabla u_i^m\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))^d} \leqslant \delta \|u_i^m\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))} + C. \quad (14)$$

For $\delta < \alpha$, applying Grönwall inequality, we obtain

$$\|u_i^m(t)\|^2_{L^2(\Omega_i^\varepsilon)} + \frac{C}{\delta} \leqslant \left(\|u_i^{\rm in}\|^2_{L^2(\Omega_i^\varepsilon)} + \frac{C}{\delta} \right) e^{2\delta t} \quad \text{for} \quad t \in (0,T), \quad (15)$$

and taking in (14) the supremum over $t \in (0,T)$, we conclude

$$\|u_i^m\|^2_{L^\infty(0,T;L^2(\Omega_i^\varepsilon))} + \|u_i^m\|^2_{L^2(0,T;H^1(\Omega_i^\varepsilon))} \leqslant C_1 \|u_i^{\rm in}\|^2_{L^2(\Omega_i^\varepsilon)} + C_2 K_{\rm g}^2 + C_3, \quad C_1, C_2, C_3 \geqslant 0.$$

Hence, using (6) from (12), it follows $\|\partial_t u_i^m\|^2_{L^2(0,T;H^1(\Omega_i^\varepsilon)^*)} = O(1)$ uniformly with respect to $m \to \infty$ and $\varepsilon \to 0$, and the continuous embedding of the solution in $C(0,T;L^2(\Omega_i^\varepsilon))$ holds; see Dautray and Lions.[39, p509]
Therefore, the mapping $\mathcal{M} : \mathcal{U}_i^\varepsilon \mapsto \mathcal{U}_i^\varepsilon$ defined when solving (11) has compact image, and hence, there exists an accumulation point $u_i^\varepsilon \in \mathcal{U}_i^\varepsilon$, $i = 1, 2$, and a subsequence still denoted by m such that as $m \to \infty$

$$u_i^m \rightharpoonup u_i^\varepsilon \text{ weakly in } \quad \mathcal{U}_i^\varepsilon \quad \text{and} \quad u_i^m \to u_i^\varepsilon \text{ strongly in } \quad L^2(0,T;L^2(\Gamma^\varepsilon)).$$

The continuity of $\mathcal{M}$ in the weak topology is justified using the Lipschitz continuity of the nonlinear term g_i in (7). Applying the fixed point theorem[40, section 4.8, theorem 8.1, p293] and the a priori estimate (9) proves the existence of a weak solution of problem (8).
To prove uniqueness, we consider the difference $w_i^\varepsilon := u_i^{1,\varepsilon} - u_i^{2,\varepsilon}$, $i = 1, 2$, of two solutions of (8) with the test function $v_i = w_i^\varepsilon$:

$$\frac{1}{2}\int_{\Omega_i^\varepsilon} (w_i^\varepsilon)^2 \Big|_{t=0}^T \, dx + \int_0^T \int_{\Omega_i^\varepsilon} A_i^\varepsilon \nabla w_i^\varepsilon \cdot \nabla w_i^\varepsilon \, dx \, dt = I_{g_i}^\varepsilon, \quad I_{g_i}^\varepsilon := \int_0^T \int_{\Gamma^\varepsilon} \varepsilon \left(g_i(u_1^{1,\varepsilon}, u_2^{1,\varepsilon}) - g_i(u_1^{2,\varepsilon}, u_2^{2,\varepsilon}) \right) w_i^\varepsilon \, d\sigma_x \, dt. \quad (16)$$

The integral $I_{g_i}^\varepsilon$ is estimated due to the Lipschitz continuity (7) as

$$|I_{g_i}^\varepsilon| \leqslant \varepsilon L_{\rm g} \int_0^T \int_{\Gamma^\varepsilon} (|w_1^\varepsilon|^2 + |w_2^\varepsilon|^2) w_i^\varepsilon \, d\sigma_x \, dt. \quad (17)$$

Then, collecting the expressions (16) and (17), applying the Cauchy-Schwarz and Grönwall inequalities, we get

$$\sum_{i=1}^2 \|w_i^\varepsilon(t)\|^2 \leqslant \sum_{i=1}^2 \|w_i^\varepsilon(0)\|^2 e^{4K_{\rm tr}L_{\rm g}t} = 0$$

and hence conclude $w_i^\varepsilon \equiv 0$, which proves $u_i^{1,\varepsilon} \equiv u_i^{2,\varepsilon}$.

(ii) To prove the nonnegativity of u_i^ε, we decompose the solution into the positive and the negative parts as: $u_i^\varepsilon = (u_i^\varepsilon)^+ - (u_i^\varepsilon)^-$ and substitute it in the Equation (8) with the test function $v_i = (u_i^\varepsilon)^-$. The assumption of the positive production rate (10) together with the uniform ellipticity (5) of A_i^ε and the nonnegativity of the initial data lead to the estimate:

$$\sup_{t\in(0,T)} \frac{1}{2}\int_{\Omega_i^\varepsilon} ((u_i^\varepsilon)^-)^2\, dx + \alpha\|\nabla(u_i^\varepsilon)^-\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))^d} \leqslant \frac{1}{2}\int_{\Omega_i^\varepsilon} ((u_i^\varepsilon)^-)^2\Big|_{t=0}\, dx = 0;$$

hence, $(u_i^\varepsilon)^- \equiv 0$ and $u_i^\varepsilon \geqslant 0$. If $u_i^\varepsilon(0) = u_i^{in} > 0$ everywhere in $\overline{\Omega}$, then $u_i^\varepsilon(t) > 0$ at least for t sufficiently small, which follows by the continuity of the solution. This completes the proof. □

We note that Theorem 1 can be extended for inhomogeneous diffusion equations (4a), where the uniform upper bound is proved in Gurevich and Reichelt[41] for reaction functions distributed over domains Ω_i^ε.

4 | PERIODIC UNFOLDING TECHNIQUE

Following Cioranescu et al,[42] we recall the technique based on the periodic unfolding and averaging operators providing continuous mappings between the components $\overline{\Omega_i^\varepsilon}$ and $\overline{Y_i}$, i = 1,2, up to the boundaries.

Definition 1. For $u(x) \in L^2(\Omega_i^\varepsilon)$, the unfolding operator $T_\varepsilon : L^2(\Omega_i^\varepsilon) \mapsto L^2(\Omega; L^2(Y_i))$, $i = 1, 2$, in the domain is defined by

$$(T_\varepsilon u)(x, y) := u\left(\varepsilon\left\lfloor\frac{x}{\varepsilon}\right\rfloor + \varepsilon y\right), \quad \text{for} \quad x \in \Omega \quad \text{and} \quad y \in Y_i, \tag{18a}$$

and for $u(x) \in L^2(\partial\Omega_i^\varepsilon)$, the operator $T_\varepsilon : L^2(\partial\Omega_i^\varepsilon) \mapsto L^2(\Omega; L^2(\partial Y_i))$, $i = 1, 2$, is performed on the boundary by

$$(T_\varepsilon u)(x, y) := u\left(\varepsilon\left\lfloor\frac{x}{\varepsilon}\right\rfloor + \varepsilon y\right), \quad \text{for} \quad x \in \Omega \quad \text{and} \quad y \in \partial Y_i. \tag{18b}$$

For $\varphi(x, y) \in L^2(\Omega; L^2(Y_i))$, the averaging operator $T_\varepsilon^{-1} : L^2(\Omega; L^2(Y_i)) \mapsto L^2(\Omega_i^\varepsilon)$, $i = 1, 2$, in the domain is defined by

$$(T_\varepsilon^{-1}\varphi)(x) := \frac{1}{|Y|}\int_{Y_i} \varphi\left(\varepsilon\left\lfloor\frac{x}{\varepsilon}\right\rfloor + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}\right)\, dz, \quad \text{for} \quad x \in \Omega_i^\varepsilon, \tag{19a}$$

and for $\varphi(x, y) \in L^2(\Omega; L^2(\partial Y_i))$, the operator $T_\varepsilon^{-1} : L^2(\Omega; L^2(\partial Y_i)) \mapsto L^2(\partial\Omega_i^\varepsilon)$, $i = 1, 2$, on the boundary is expressed by

$$(T_\varepsilon^{-1}\varphi)(x) := \frac{1}{|Y|}\int_{Y_i} \varphi\left(\varepsilon\left\lfloor\frac{x}{\varepsilon}\right\rfloor + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}\right)\, dz, \quad \text{for} \quad x \in \partial\Omega_i^\varepsilon. \tag{19b}$$

Abusing the notation T_ε^{-1} is used for a left inverse operator of T_ε according to Lemma 1 (i), which is also right inverse in the special cases accounting in Lemma 1 (ii). For those functions that belong to $H^1(\Omega_i^\varepsilon)$, the restriction of the unfolding operator T_ε is well-defined as the mapping $H^1(\Omega_i^\varepsilon) \mapsto L^2(\Omega; H^1(Y_i))$, and for functions in $L^2(\Omega; H^1(Y_i))$, the restriction of the averaging operator T_ε^{-1} is well-defined as $L^2(\Omega; H^1(Y_i)) \mapsto H^1(\bigcup_{\lambda\in I_\varepsilon} Y_i^\lambda)$, where Y_i^λ is from (3). We note that the spaces $H^1(\bigcup_{\lambda\in I_\varepsilon} Y_i^\lambda)$ and $H^1(\Omega_i^\varepsilon)$ do not coincide because functions from $H^1(\bigcup_{\lambda\in I_\varepsilon} Y_i^\lambda)$ are discontinuous while they can have jumps across the interface Γ^ε.

The operator properties are collected below in Lemma 1.

Lemma 1 (Properties of the operators T_ε and T_ε^{-1}). *For arbitrary* $x \mapsto u(x) \in H^1(\Omega_i^\varepsilon) \cap L^2(\partial\Omega_i^\varepsilon)$ *and* $(x, y) \mapsto \varphi(x, y) \in L^2(\Omega; H^1(Y_i) \cap L^2(\partial Y_i))$, $i = 1, 2$, *and the extension by zero:* $\bar{u}(x) = u(x)$ *for* $x \in \Omega_i^\varepsilon$, *otherwise* $\bar{u}(x) = 0$ *for* $x \in \Omega \setminus \overline{\Omega_i^\varepsilon}$, *the following properties hold:*

(i) invertibility of T_ε: $(T_\varepsilon^{-1}T_\varepsilon)u(x) = u(x)$;
(ii) invertibility of T_ε^{-1}:

(iia) $(T_\varepsilon T_\varepsilon^{-1}\varphi)(x, y) = \varphi(y)$ *for* $x \in \Omega$, *if* $\varphi(y)$ *is a constant or periodic function of the argument* $y \in Y_i$,
(iib) $(T_\varepsilon T_\varepsilon^{-1}\bar{u})(x, \cdot) = (T_\varepsilon^{-1}\bar{u})(x) = \frac{|Y_i|}{|Y|}\langle T_\varepsilon u\rangle_{Y_i}(x)$ *for* $x \in \Omega$, *where is the average* $\langle\cdot\rangle_{Y_i} = \frac{1}{|Y_i|}\int_{Y_i}(\cdot)\, dy$;

(iii) composition rule: $T_\varepsilon(\mathcal{F}(u))(x, y) = \mathcal{F}(T_\varepsilon u)(x, y)$ *for any elementary function* $\mathcal{F}$;
(iv) chain rules: $\varepsilon T_\varepsilon(\nabla u)(x, y) = \nabla_y(T_\varepsilon u)(x, y)$, *and* $\nabla(T_\varepsilon^{-1}\varphi)(x) = T_\varepsilon^{-1}(\nabla\varphi + \frac{1}{\varepsilon}\nabla_y\varphi)(x)$ *for* $x \in Y_i^\lambda$ *and* $\varphi \in H^1(\Omega \times Y_i)$;

(v) integration rules:

$$\int_{\Omega_i^\varepsilon} u(x)\,dx = \frac{1}{|Y|}\int_{\Omega\times Y_i} (T_\varepsilon u)(x,y)\,dx\,dy, \tag{20a}$$

$$\int_{\partial\Omega_i^\varepsilon} u(x)\,d\sigma_x = \frac{1}{\varepsilon|Y|}\int_{\Omega\times\partial Y_i} (T_\varepsilon u)(x,y)\,dx\,d\sigma_y; \tag{20b}$$

(vi) boundedness of T_ε:

$$\int_{\Omega_i^\varepsilon} u^2(x)\,dx = \frac{1}{|Y|}\int_{\Omega\times Y_i} (T_\varepsilon u)^2(x,y)\,dx\,dy, \tag{21a}$$

$$\int_{\Omega_i^\varepsilon} |\nabla u|^2(x)\,dx = \frac{1}{\varepsilon^2|Y|}\int_{\Omega\times Y_i} |\nabla_y(T_\varepsilon u)|^2(x,y)\,dx\,dy, \tag{21b}$$

$$\int_{\partial\Omega_i^\varepsilon} u^2(x)\,d\sigma_x = \frac{1}{\varepsilon|Y|}\int_{\Omega\times\partial Y_i} (T_\varepsilon u)^2(x,y)\,dx\,d\sigma_y. \tag{21c}$$

Proof. The property (iib) follows in a straightforward manner from the calculation of $(T_\varepsilon T_\varepsilon^{-1}\bar{u})(x,z) = (T_\varepsilon^{-1}\bar{u})(x)$ for $x \in \Omega$ and $z \in Y$:

$$\frac{1}{|Y|}\int_{Y_i} \bar{u}\left(\varepsilon\left\lfloor \frac{\varepsilon\left\lfloor\frac{x}{\varepsilon}\right\rfloor + \varepsilon z}{\varepsilon}\right\rfloor + \varepsilon y\right)\,dy = \frac{1}{|Y|}\int_{Y_i} \bar{u}\left(\varepsilon\left\lfloor\frac{x}{\varepsilon}\right\rfloor + \varepsilon y\right)\,dy = T_\varepsilon^{-1}\bar{u}(x)$$

and the fact that $T_\varepsilon^{-1}\bar{u} = \frac{|Y_i|}{|Y|}\langle T_\varepsilon u\rangle_{Y_i}$ as a consequence of the definition (19a) if $\varphi(x,y) \equiv \bar{u}(x)$ for all $\varphi(x,y) \in L^2(\Omega; H^1(Y_i))$. The proof of the other properties can be found in other studies.[20,21,31,42,43] □

5 | ASYMPTOTIC ANALYSIS

In this section, we collect some auxiliary tools used later in the derivation of the residual error estimates.

Lemma 2 (Poincaré inequality in periodic domains). *For $u(x) \in H^1(\Omega_i^\varepsilon)$, the following Poincaré inequality holds (see, eg, Cioranescu et al[42,43]):*

$$\|u - \langle T_\varepsilon u\rangle_{Y_i}\|^2_{L^2(\Omega_i^\varepsilon)} \leqslant \varepsilon^2 K_{\mathrm{P}}\|\nabla u\|^2_{L^2(\Omega_i^\varepsilon)^d}, \quad K_{\mathrm{P}} > 0. \tag{22}$$

Proof. We recall the Poincaré inequality for a function $\varphi(y) \in H^1(Y_i)$ in the unit cell with connected subsets Y_i for $i = 1, 2$:

$$\int_{Y_i} (\varphi - \langle\varphi\rangle_{Y_i})^2\,dy \leqslant K_{\mathrm{P}}\int_{Y_i} |\nabla_y\varphi|^2\,dy, \quad \langle\varphi\rangle_{Y_i} := \frac{1}{|Y_i|}\int_{Y_i}\varphi(y)\,dy. \tag{23}$$

Integrating (23) over Ω yields

$$\int_{\Omega\times Y_i} |\varphi - \langle\varphi\rangle_{Y_i}|^2\,dx\,dy \leqslant K_{\mathrm{P}}\int_{\Omega\times Y_i} |\nabla_y\varphi|^2\,dx\,dy$$

for all $\varphi \in L^2(\Omega; H^1(Y_i))$. Choosing $\varphi = T_\varepsilon u$ gives

$$\begin{aligned}\frac{1}{|Y|}\int_{\Omega\times Y_i} |T_\varepsilon u - \langle T_\varepsilon u\rangle_{Y_i}|^2\,dx\,dy &\leqslant \frac{K_{\mathrm{P}}}{|Y|}\int_{\Omega\times Y_i} |\nabla_y(T_\varepsilon u)|^2\,dx\,dy \\ &= K_{\mathrm{P}}\varepsilon^2\|\nabla u\|^2_{L^2(\Omega_i^\varepsilon)}.\end{aligned}$$

For the left-hand side, we use the composition rule (iii) as well as $T_\varepsilon\langle T_\varepsilon u\rangle_{Y_i} = \langle T_\varepsilon u\rangle_{Y_i}$. For all $(x,y) \in \Omega \times Y_i$, we have

$$\begin{aligned}&\left(T_\varepsilon\langle T_\varepsilon u\rangle_{Y_i}\right)(x,y) = \left(T_\varepsilon\left((x,z) \mapsto \frac{1}{|Y|}\int_{Y_i} u\left(\varepsilon\left\lfloor\frac{x}{\varepsilon}\right\rfloor + \varepsilon z\right)\,dz\right)\right)(x,y) \\ &= \frac{1}{|Y|}\int_{Y_i} u\left(\varepsilon\left\lfloor \frac{\varepsilon\left\lfloor\frac{x}{\varepsilon}\right\rfloor + \varepsilon y}{\varepsilon}\right\rfloor + \varepsilon z\right)\,dz = \frac{1}{|Y|}\int_{Y_i} u\left(\varepsilon\left\lfloor\frac{x}{\varepsilon}\right\rfloor + \varepsilon z\right)\,dz = \langle T_\varepsilon u\rangle_{Y_i}(x),\end{aligned}$$

while noting that $\left\lfloor \frac{\varepsilon\left\lfloor \frac{x}{\varepsilon}\right\rfloor+\varepsilon y}{\varepsilon} \right\rfloor = \left\lfloor \frac{x}{\varepsilon}\right\rfloor$ for all $y \in (0,1)^d$. This shows, in particular, that $y \mapsto \left(T_\varepsilon \langle T_\varepsilon u\rangle_{Y_i}\right)(x,y)$ is constant for a.e. $x \in \Omega$. □

We recall the trace theorem in unit cells for a function $\varphi \in L^2(\Omega; H^1(Y_i))$:

$$\|\varphi\|^2_{L^2(\partial Y_i)} \leqslant K_{\mathrm{tr}}(\|\varphi\|^2_{L^2(Y_i)} + \|\nabla_y \varphi\|^2_{L^2(Y_i)^d}) = K_{\mathrm{tr}}\|\varphi\|^2_{H^1(Y_i)}, \tag{24}$$

with $K_{\mathrm{tr}} > 0$. After the substitution of $\varphi = T_\varepsilon u$ for the function $u(x) \in H^1(\Omega_i^\varepsilon)$, there follows (see, eg, Monsurrò[44]):

$$\|u\|^2_{L^2(\partial\Omega_i^\varepsilon)} \leqslant K_{\mathrm{tr}}\left(\frac{1}{\varepsilon}\|u\|^2_{L^2(\Omega_i^\varepsilon)} + \varepsilon\|\nabla u\|^2_{L^2(\Omega_i^\varepsilon)^d}\right). \tag{25}$$

In particular, repeating the arguments in the proof of Lemma 2, the trace inequality in periodic domains can be shown:

$$\|u - \langle T_\varepsilon u\rangle_{Y_i}\|^2_{L^2(\partial\Omega_i^\varepsilon)} \leqslant \varepsilon K_{\mathrm{tr}}(1 + K_{\mathrm{P}})\|\nabla u\|^2_{L^2(\Omega_i^\varepsilon)^d}. \tag{26}$$

Lemma 3 (Uniform extension in connected periodic domains). *For $u(x) \in H^1(\Omega_i^\varepsilon)$, there exists a continuous extension $\tilde{u} \in H^1(\Omega)$ from the connected set Ω_i^ε to Ω such that $\tilde{u} = u$ in Ω_i^ε and*

$$\|\tilde{u}\|^2_{L^2(\Omega)} \leqslant K_{\mathrm{e}}\|u\|^2_{L^2(\Omega_i^\varepsilon)}, \quad \|\nabla\tilde{u}\|^2_{L^2(\Omega)^d} \leqslant K_{\mathrm{e}}\|\nabla u\|^2_{L^2(\Omega_i^\varepsilon)^d}, \quad K_{\mathrm{e}} > 0. \tag{27}$$

If $u = 0$ on $\partial\Omega_i^\varepsilon \cap \partial\Omega$, then $\tilde{u} \in H_0^1(\Omega)$ exists satisfying (27).

Proof. Indeed, the assertion holds in accordance with previous studies,[3,4,45, chapter 4] and the zero trace at the boundary $\partial\Omega$ is argumented in Höpker.[46, theorem 3.5] □

Below, we recall the auxiliary result from Fellner and Kovtunenko[20, lemma 2] and Kovtunenko and Zubkova.[21, lemma 4.1]

Lemma 4 (Asymptotic restriction from Ω to Ω_i^ε). *For given functions $u, v \in H^1(\Omega)$ (which have no jumps across the interface Γ^ε), the asymptotic estimate*

$$\left|\int_{\Omega_i^\varepsilon} uv\, dx - \frac{|Y_i|}{|Y|}\int_\Omega uv\, dx\right| \leqslant \varepsilon K_{\mathrm{r}}\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \quad K_{\mathrm{r}} > 0, \tag{28}$$

holds as $\varepsilon \to 0$ for $i = 1, 2$.

Based on the geometric assumptions (D1) to (D4), we define the space of periodic functions in the cells Y_i by

$$H^1_\#(Y_i) := \{\varphi \in H^1(Y_i): \quad \varphi(y)|_{y_j=0} = \varphi(y)|_{y_j=1},\ j = 1, \dots, d, \quad \text{for} \quad y \in \partial Y_i \cap \partial Y\}. \tag{29}$$

We set the standard cell problem determining $N^i = (N_1^i, \dots, N_d^i)(y)$, $i = 1, 2$, from

$$\mathrm{div}_y\left(A_i(\partial_y N^i + I)\right) = 0 \quad \text{in} \ \ Y_i, \tag{30a}$$

$$A_i(\partial_y N^i + I)n_i = 0 \quad \text{on} \quad \Gamma, \tag{30b}$$

$$(\partial_y N^i + I)A_i|_{y_k=0} = (\partial_y N^i + I)A_i\Big|_{y_k=1}, \quad N^i|_{y_k=0} = N^i|_{y_k=1} \quad \text{for} \quad k = 1, \dots, d, \tag{30c}$$

where the last line in (30c) implies that $N_k^i \in H^1_\#(Y_i)$ for $i = 1, 2$ and $k = 1, \dots, d$. In (30), the notation $\partial_y N^i(y) \in \mathbb{R}^{d\times d}$ for $y \in Y_i$ stands for the matrix of derivatives with entries $(\partial_y N^i(y))_{kl} = \frac{\partial N_k^i}{\partial y_l}$, $k, l = 1, \dots, d$, and $I \in \mathbb{R}^{d\times d}$ denotes the identity matrix. The system (30) admits the weak formulation: find vector-functions $N^i \in H^1_\#(Y_i)^d$ such that

$$\int_{Y_i} A_i(\partial_y N^i + I)\nabla_y \varphi\, dy = 0, \tag{31}$$

for all test functions $\varphi \in H^1_\#(Y_i)$. A solution of (31) exists, and it is defined up to a constant in Y_i.

Based on the solution N^i of the cell problem (31), the diffusivity matrices A_i admit the following asymptotic representation formulated in the lemma below; see Fellner and Kovtunenko[20] and Kovtunenko and Zubkova.[21]

Lemma 5 (Asymptotic formula for periodic diffusivity matrices)**.**

(i) For the solution N^i of the cell problem (31), the following representation holds:

$$A_i(y)(\partial_y N^i(y) + I) = A_i^0 + B_i(y), \tag{32}$$

with $A_i^0 \in \mathbb{R}^{d\times d}_{sym}$ given by the averaging

$$A_i^0 := \langle A_i(\partial_y N^i + I)\rangle_{Y_i}, \tag{33}$$

and it is a symmetric d-by-d matrix:

$$\textit{There exists} \quad \underline{a}^0 \geqslant 0 \quad \textit{such that} \quad \xi^\top A^0\ \xi \geqslant \underline{a}^0 |\xi|^2 \textit{ for } \quad \xi \in \mathbb{R}^d. \tag{34}$$

The d-by-d matrix $B_i(y)$ is periodic and has the following divergence form in the cell Y_i:

$$(B_i)_{kl} = \sum_{m=1}^d b^{(i)}_{klm,m}, \quad k,l = 1, \dots, d, \quad \textit{where} \quad b^{(i)}_{klm,m} = \frac{\partial b^{(i)}_{klm}}{\partial y_m}.$$

Its components $b^{(i)}_{klm}$ are skew-symmetric:

$$b^{(i)}_{klm} + b^{(i)}_{kml} = 0, \quad k,l,m = 1, \dots, d,$$

the matrix B_i is divergence-free:

$$\sum_{l,m=1}^d b^{(i)}_{klm,lm} = 0 \quad \textit{with} \quad b^{(i)}_{klm,lm} = \frac{\partial^2 b^{(i)}_{klm}}{\partial y_l \partial y_m},$$

and the average $\langle B_i\rangle_{Y_i} = 0$. At the interface, the condition holds:

$$(A_i^0 + B_i)n_i = 0 \quad \textit{on} \quad \Gamma. \tag{35}$$

(ii) Assume that $N^i \in W^{1,\infty}(Y_i)^d$. For varying function $v_i \in \mathcal{V}_i^\varepsilon$ and fixed $u_i^0 \in L^2(0,T;H^3(\Omega))$, the following integral form corresponding to the averaged equation (50):

$$I_{A_i^0} := \int_{\Omega_i^\varepsilon} A_i^0 \nabla u_i^0 \cdot \nabla v_i \, dx - \int_{\Gamma^\varepsilon} A_i^0 \nabla u_i^0 \cdot n_i v_i \, d\sigma_x \tag{36}$$

with the help of the corrector $u_i^1 := u_i^0 + \varepsilon(T_\varepsilon^{-1} N^i) \cdot \nabla u_i^0$ is approximated as follows:

$$Err_0(v_i, \varepsilon) := \int_0^T \left(I_{A_i^0} - \int_{\Omega_i^\varepsilon} A_i^\varepsilon \nabla u_i^1 \cdot \nabla v_i \, dx \right) dt,$$

$$|Err_0(v_i,\varepsilon)| \leqslant \varepsilon K \|A_i\|_{L^\infty(Y_i)} \left(\|N^i\|_{L^\infty(Y_i)^d} + \|\partial_y N^i\|_{L^\infty(Y_i)^{d\times d}} + 1 \right) \|u_i^0\|_{L^2(0,T;H^3(\Omega_i^\varepsilon))} \|v_i\|_{L^2(0,T;H^1(\Omega_i^\varepsilon))}, \quad K > 0. \tag{37}$$

Proof.

(i) For the vector-valued solution N_i of (31), the representation (32) follows from the Helmholtz theorem; see Zhikov et al.[36, section 1.1] The interface condition (35) is obtained after substitution of (32) into (30b).

(ii) Let $v_i \in \mathcal{V}_i^\varepsilon$ and $u_i^0 \in L^2(0,T;H^3(\Omega))$ be given. To prove (37), we rewrite $I_{A_i^0}$ in (36) in virtue of the integration rules from Lemma 1 in the microvariable y:

$$I_{A_i^0} = \frac{1}{\varepsilon^2|Y|}\left\{\int_{\Omega\times Y_i}(T_\varepsilon A_i^0)\nabla_y(T_\varepsilon u_i^0)\cdot\nabla_y(T_\varepsilon v_i)\,dx\,dy - \int_{\Omega\times\Gamma}(T_\varepsilon A_i^0)\nabla_y(T_\varepsilon u_i^0)\cdot n_i(T_\varepsilon v_i)\,dx\,d\sigma_y\right\}. \tag{38}$$

For the constant matrix, the identity $A_i^0 = T_\varepsilon A_i^0$ holds. Then, expressing A_i^0 from (32), using the product rule

$$\partial_y N^i \nabla_y(T_\varepsilon u_i^0) = \nabla_y(N^i\cdot\nabla_y(T_\varepsilon u_i^0)) - \partial_y(\nabla_y(T_\varepsilon u_i^0))N^i,$$

the chain rule $\varepsilon T_\varepsilon(\nabla u_i^0) = \nabla_y(T_\varepsilon u_i^0)$, and the notation of the corrector $u_i^1 := u_i^0 + \varepsilon(T_\varepsilon^{-1}N^i)\cdot\nabla u_i^0$, we rearrange the following terms:

$$(T_\varepsilon A_i^0)\nabla_y(T_\varepsilon u_i^0) = (A_i + A_i(\partial_y N^i) - B_i)\nabla_y(T_\varepsilon u_i^0) = A_i\nabla_y(T_\varepsilon u_i^1) - A_i\partial_y(\nabla_y(T_\varepsilon u_i^0))N^i - B_i\nabla_y(T_\varepsilon u_i^0).$$

Taking into account this formula, $I_{A_i^0}$ is performed equivalently by

$$\begin{aligned} I_{A_i^0} = \frac{1}{\varepsilon^2|Y|}\Bigg\{&\int_{\Omega\times Y_i}\left[A_i\nabla_y(T_\varepsilon u_i^1))\cdot\nabla_y(T_\varepsilon v_i) - A_i\partial_y(\nabla_y(T_\varepsilon u_i^0))N^i\cdot\nabla_y(T_\varepsilon v_i)\right]\,dx\,dy \\ &-\int_{\Omega\times\Gamma}A_i^0\nabla_y(T_\varepsilon u_i^0)\cdot n_i(T_\varepsilon v_i)\,dx\,d\sigma_y\Bigg\} + I_{B_i}, \end{aligned} \tag{39}$$

with the integral I_{B_i} is written component-wisely as follows:

$$I_{B_i} := -\frac{1}{\varepsilon^2|Y|}\int_{\Omega\times Y_i}B_i\nabla_y(T_\varepsilon u_i^0)\cdot\nabla_y(T_\varepsilon v_i)\,dx\,dy = -\frac{1}{\varepsilon^2|Y|}\int_{\Omega\times Y_i}\sum_{k,l,m=1}^d b_{klm,m}^{(i)}(T_\varepsilon u_i^0)_{,k}(T_\varepsilon v_i)_{,l}\,dx\,dy.$$

Recalling the definition of B_i and the fact that it is divergence-free, the term I_{B_i} is integrated by parts as follows:

$$I_{B_i} = \frac{1}{\varepsilon^2|Y|}\int_{\Omega\times Y_i}\sum_{k,l,m=1}^d b_{klm,m}^{(i)}(T_\varepsilon u_i^0)_{,kl}(T_\varepsilon v_i)\,dx\,dy - \frac{1}{\varepsilon^2|Y|}\int_{\Omega\times\partial Y_i}B_i\nabla_y(T_\varepsilon u_i^0)\cdot n_i(T_\varepsilon v_i)\,dx\,d\sigma_y. \tag{40}$$

After substitution of (40) in (39), the integral over Γ disappears due to the interface condition (35). The integral over $\partial Y_i\setminus\Gamma$ vanishes after rewriting the integral again in macrovariables because of $v_i = 0$ on $\partial\Omega_i^\varepsilon\cap\partial\Omega$ and because jumps across the cell boundary of v_i and ∇u_i^0 are zero (by assumed H^3-, hence, C^1-smoothness of u_i^0), while B_i is periodic.

The integral over $\Omega\times Y_i$ in (40) can be rewritten using the zero average $\langle B_i\rangle_{Y_i} = 0$ as follows:

$$\frac{1}{\varepsilon^2|Y|}\int_{\Omega\times Y_i}\sum_{k,l,m=1}^d b_{klm,m}^{(i)}(T_\varepsilon u_i^0)_{,kl}(T_\varepsilon v_i)\,dx\,dy = I_1^i + I_2^i,$$

where

$$I_1^i := \frac{1}{\varepsilon^2|Y|}\int_{\Omega\times Y_i}\sum_{k,l,m=1}^d b_{klm,m}^{(i)}(T_\varepsilon u_i^0)_{,kl}(T_\varepsilon v_i - \langle T_\varepsilon v_i\rangle_{Y_i})\,dx\,dy,$$

$$I_2^i := \frac{1}{\varepsilon^2|Y|}\int_{\Omega\times Y_i}\langle T_\varepsilon v_i\rangle_{Y_i}\sum_{k,l,m=1}^d b_{klm,m}^{(i)}[(T_\varepsilon u_i^0)_{,kl} - \langle(T_\varepsilon u_i^0)_{,kl}\rangle_{Y_i}]\,dx\,dy.$$

We rewrite I_1^i and I_2^i in the macrovariable x in all local cells using the integration rules (20) and (21) and then apply to the result the Cauchy-Schwarz inequality and the Poincaré inequality (23).

Below, the indices k, l, m will refer to both x as well as y coordinates. We are starting from

$$I_1^i = \frac{1}{\varepsilon^2|Y|}\int_{\Omega\times Y_i}\sum_{k,l,m=1}^{d} T_\varepsilon(T_\varepsilon^{-1}b_{klm,m}^{(i)})(T_\varepsilon u_i^0)_{,kl}T_\varepsilon(v_i - \langle T_\varepsilon v_i\rangle_{Y_i})\ dx\ dy = \int_{\Omega_i^\varepsilon}\sum_{k,l,m=1}^{d}\varepsilon\left(T_\varepsilon^{-1}b_{klm}^{(i)}\right)_{,m}u_{i,kl}^0(v_i - \langle v_i\rangle_{Y_i^\lambda})\ dx,$$

where it is for all $x \in \Omega_i^\varepsilon$:

$$\langle T_\varepsilon v_i\rangle_{Y_i}(x) = \frac{1}{|Y_i|}\int_{Y_i} v_i\left(\varepsilon\left\lfloor\frac{x}{\varepsilon}\right\rfloor + \varepsilon z\right)\ dz = \frac{1}{|\varepsilon(\lambda + Y_i)|}\int_{\varepsilon(\lambda+Y_i)} v_i(z)\ dz = \langle v_i\rangle_{Y_i^\lambda}(x)$$

with $\lambda = \lfloor\frac{x}{\varepsilon}\rfloor$. First, there are some constants $0 < K_1 \leqslant K_2$ such that

$$\begin{aligned}|I_1^i| &= \left|\sum_{\lambda\in I_\varepsilon}\int_{Y_i^\lambda}\sum_{k,l,m=1}^{d}(\varepsilon T_\varepsilon^{-1}b_{klm,m}^{(i)})u_{i,kl}^0(v_i - \langle v_i\rangle_{Y_i^\lambda})\ dx\right| \\ &\leqslant K_1\|B_i\|_{L^\infty(Y_i)^{d\times d}}\|u_i^0\|_{H^2(\Omega_i^\varepsilon)}\varepsilon\|\nabla v_i\|_{L^2(\Omega_i^\varepsilon)^d} \leqslant \varepsilon K_2(\|A_i\|_{L^\infty(Y_i)^{d\times d}}\|\partial_y N^i\|_{L^\infty(Y_i)^{d\times d}} + 1)\|u_i^0\|_{H^2(\Omega_i^\varepsilon)}\|\nabla v_i\|_{L^2(\Omega_i^\varepsilon)^d}.\end{aligned} \tag{41}$$

Similarly, there exists $K_3 > 0$ such that

$$|I_2^i| \leqslant K_3(\|A_i\|_{L^\infty(Y_i)^{d\times d}}\|\partial_y N^i\|_{L^\infty(Y_i)^{d\times d}} + 1)\sum_{k,l=1}^{d}\varepsilon\|\nabla(u_{i,kl}^0)\|_{L^2(\Omega_i^\varepsilon)^d}\|v_i\|_{L^2(\Omega_i^\varepsilon)}. \tag{42}$$

We substitute in (39) the expression of I_{B_1} from (40) and use (35), such that

$$I_{A_i^0} - \frac{1}{\varepsilon^2|Y|}\int_{\Omega\times Y_i} A_i\nabla_y(T_\varepsilon u_i^1)\cdot\nabla_y(T_\varepsilon v_i)\ dx\ dy = \frac{1}{\varepsilon^2|Y|}\int_{\Omega\times Y_i} A_i\partial_y(\nabla_y(T^\varepsilon u_i^0))N^i\cdot\nabla_y(T_\varepsilon v_i)\ dx\ dy + I_1^i + I_2^i. \tag{43}$$

Rewriting the integrals in microvariables with the help of the integration rules (20) and (21), the following estimate takes place with $K_4 > 0$:

$$\left|I_{A_i^0} - \int_{\Omega_i^\varepsilon} A_i^\varepsilon\nabla u_i^1\cdot\nabla v_i\ dx\right| \leqslant |I_1^i| + |I_2^i| + \varepsilon K_4\|A_i\|_{L^\infty(Y_i)^{d\times d}}\|N^i\|_{L^\infty(Y_i)^d}\|u_i^0\|_{H^2(\Omega_i^\varepsilon)}\|\nabla v_i\|_{L^2(\Omega_i^\varepsilon)^d}. \tag{44}$$

Using the estimates (41) and (42), from (44) after integration over time, it follows (37) that proves the assertion of Lemma 5. □

With these preliminaries, in the next section, we homogenize the nonlinear transmission problem (8) as $\varepsilon \to 0$.

6 | THE MAIN HOMOGENIZATION RESULT

We state the averaged bidomain diffusion problem determining the functions $u_i^0(t, x)$, $i = 1, 2$, in the time-space domain $(0, T)\times\Omega$ from

$$\partial_t u_i^0 - \operatorname{div}(A_i^0\nabla u_i^0) = \frac{|\Gamma|}{|Y_i|}g_i(u_1^0, u_2^0) \quad \text{in} \quad \Omega, \tag{45a}$$

$$u_i^0 = 0 \quad \text{on} \quad \partial\Omega, \tag{45b}$$

$$u_i^0 = u_i^{\text{in}} \quad \text{as} \quad t = 0, \tag{45c}$$

where the effective matrices A_i^0 are defined in (33). It implies the variational formulation: find $u_i^0 \in \mathcal{V}^0$ in the space

$$\mathcal{V}^0 = \{u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) : \quad \partial_t u \in L^2(0, T; H^1(\Omega)^*),\ u = 0 \quad \text{on} \quad \partial\Omega\},$$

such that it satisfies the initial condition (45c) and the following nonlinear equation:

$$\int_0^T \left(\langle \partial_t u_i^0, v \rangle_\Omega + \int_\Omega \left(A_i^0 \nabla u_i^0 \cdot \nabla v - \frac{|\Gamma|}{|Y_i|} g_i(u_1^0, u_2^0) v \right) dx \right) dt = 0, \tag{46}$$

for all text functions $v \in \mathcal{V}^0 := L^2(0,T; H_0^1(\Omega))$. In (46), the notation $\langle \cdot, \cdot \rangle_\Omega$ implies the duality between $H^1(\Omega)$ and its topologically dual space $H^1(\Omega)^*$.

The solvability of (46) can be obtained in the same way as for (8) due to the uniform boundedness (6) and the continuity (7) of the nonlinear term g_i. Moreover, the a priori estimate like (9) holds (for $i = 1, 2$):

$$\|u_i^0\|_{\mathcal{U}^0}^2 \leqslant C_1 \|u_i^{\text{in}}\|_{L^2(\Omega)}^2 + C_2 K_g^2 + C_3.$$

In Theorem 2, we need smoothness of the macroscopic solution and the uniform boundedness of N^i and of its gradient in order to prove the residual error estimate, which is a standard assumption for cell problems; see, ie, Zhikov et al.[36, section 5.6, theorem 5.10] These assumptions might be weekend just to get a two-scale convergence to the homogenized problem.

Theorem 2 (Residual error estimate). *Let the cell problem (31) obey the Lipschitz continuous solution $N^i \in W^{1,\infty}(Y_i)$, and the macroscopic solution be such that $u_i^0 \in L^2(0,T; H^3(\Omega)) \cap L^\infty(0,T; H^1(\Omega))$, $\partial_t(\nabla u_i^0) \in L^2(0,T; L^2(\Omega_i^\varepsilon))^d$, $i = 1, 2$. Then the solution u_i^ε of the inhomogeneous problem (8) and the first-order corrector to the solution u_i^0 of the averaged problem (46) given by*

$$u_i^1 = u_i^0 + \varepsilon (T_\varepsilon^{-1} \tilde{N}^i) \cdot \nabla u_i^0 \quad in \quad \Omega, \tag{47}$$

where $\tilde{N}^i \in W^{1,\infty}(Y)$ is a periodic extension of N^i to Y, satisfy the residual error estimate:

$$\|u_i^\varepsilon - u_i^1\|_{\mathcal{U}_i^\varepsilon}^2 \leqslant Err_{12}(\varepsilon) = O(\varepsilon), \tag{48}$$

where Err_{12} is determined in (66).

Proof. We start with derivation of an asymptotic equation for the difference $u_i^\varepsilon - u_i^1$ (see (51)). Multiplying the diffusion equation (45a) with a test function $v_i \in \mathcal{V}_i^\varepsilon$, integrating it over $(0,T) \times \Omega_i^\varepsilon$, it follows the variational equation in two subdomains for $i = 1, 2$:

$$\int_0^T \left(\langle \partial_t u_i^0, v_i \rangle_{\Omega_i^\varepsilon} - \int_{\Omega_i^\varepsilon} \left(\operatorname{div}(A_i^0 \nabla u_i^0) + \frac{|\Gamma|}{|Y_i|} g_i(u_1^0, u_2^0) \right) v_i \, dx \right) dt = 0. \tag{49}$$

The integration by parts in (49) due to the Dirichlet condition (45b) leads to

$$\int_0^T \left(\langle \partial_t u_i^0, v_i \rangle_{\Omega_i^\varepsilon} + \int_{\Omega_i^\varepsilon} A_i^0 \nabla u_i^0 \cdot \nabla v_i \, dx - \int_{\Gamma^\varepsilon} A_i^0 \nabla u_i^0 \cdot n_i v_i \, d\sigma_x \right) dt = \int_0^T \int_{\Omega_i^\varepsilon} \frac{|\Gamma|}{|Y_i|} g_i(u_1^0, u_2^0) v_i \, dx \, dt. \tag{50}$$

We choose $v \in \mathcal{V}^0$ and $v_i \in \mathcal{V}_i^\varepsilon$. With a special choice of v_i, it can be equal to v. For test functions $v_i = v \in \mathcal{V}^0 \subset \mathcal{V}_i^\varepsilon$, $i = 1, 2$, we subtract (50) from the inhomogeneous equation (8):

$$\begin{aligned} &\int_0^T \left(\langle \partial_t (u_i^\varepsilon - u_i^0), v \rangle_{\Omega_i^\varepsilon} + \int_{\Omega_i^\varepsilon} \left(A_i^\varepsilon \nabla u_i^\varepsilon - A_i^0 \nabla u_i^0 \right) \cdot \nabla v \, dx + \int_{\Gamma^\varepsilon} A_i^0 \nabla u_i^0 \cdot n_i v \, d\sigma_x \right) dt \\ &= \int_0^T \left(\int_{\Gamma^\varepsilon} \varepsilon g_i(u_1^\varepsilon, u_2^\varepsilon) v \, d\sigma_x - \int_{\Omega_i^\varepsilon} \frac{|\Gamma|}{|Y_i|} g_i(u_1^0, u_2^0) v \, dx \right) dt \end{aligned}$$

and gather the terms as follows:

$$\int_0^T \left(\langle \partial_t(u_i^\varepsilon - u_i^1), v\rangle_{\Omega_i^\varepsilon} + \int_{\Omega_i^\varepsilon} A_i^\varepsilon \nabla(u_i^\varepsilon - u_i^1)\cdot \nabla v\, dx \right) dt - I_i(v) = \sum_{k=0}^{3} \mathrm{Err}_k(v,\varepsilon), \tag{51}$$

where the following notation was used

$$I_i(v) := \int_0^T \left(\int_{\Gamma^\varepsilon} \varepsilon g_i(u_1^\varepsilon, u_2^\varepsilon) v\, d\sigma_x - \frac{|\Gamma|}{|Y|}\int_\Omega g_i(u_1^1,u_2^1) v\, dx \right) dt. \tag{52}$$

Err_0 is given by the formula (37) from Lemma 5, and other residual error functions Err_k, $k = 1,2,3$, in the right-hand side of (51) will be introduced and estimated next.

We use the Cauchy-Schwarz inequality and the expansion of the time-derivative of the corrector $\partial_t u_i^1 = \partial_t[u_i^0 + \varepsilon(T_\varepsilon^{-1}N^i)\cdot\nabla u_i^0]$ implying that

$$\begin{aligned}
&\mathrm{Err}_1(v,\varepsilon) := -\int_0^T \langle \partial_t(u_i^1 - u_i^0), v\rangle_{\Omega_i^\varepsilon}\, dt,\\
&|\mathrm{Err}_1(v,\varepsilon)| \leqslant \|\partial_t u_i^1 - \partial_t u_i^0\|_{L^2(0,T;H^1(\Omega_i^\varepsilon)^*)}\|v\|_{L^2(0,T;H^1(\Omega_i^\varepsilon))} \leqslant \varepsilon\|N^i\|_{L^\infty(Y_i)^d}\|\partial_t(\nabla u_i^0)\|_{L^2(0,T;H^1(\Omega_i^\varepsilon)^*)^d}\|v\|_{L^2(0,T;H^1(\Omega_i^\varepsilon))}.
\end{aligned} \tag{53}$$

Applying to $g_i(u_1^0,u_2^0)v$ the restriction operator from Lemma 4, then using the boundedness (6) and the Lipschitz continuity (7) for g_i leads to

$$\begin{aligned}
&\mathrm{Err}_2(v,\varepsilon) := -\int_0^T \left(\frac{|\Gamma|}{|Y_i|}\int_{\Omega_i^\varepsilon} g_i(u_1^0,u_2^0) v\, dx - \frac{|\Gamma|}{|Y|}\int_\Omega g_i(u_1^0,u_2^0) v\, dx \right) dt\\
&|\mathrm{Err}_2(v,\varepsilon)| \leqslant \varepsilon K_6 K_{\mathrm{g}} \|v\|_{L^2(0,T;H^1(\Omega))}, \quad K_6 = \frac{K_{\mathrm{r}}|\Gamma|}{|Y_i|}\sqrt{T|\Omega|},
\end{aligned} \tag{54}$$

and the further error function (with $K_7 = |\Gamma| L_{\mathrm{g}}$)

$$\begin{aligned}
&\mathrm{Err}_3(v,\varepsilon) := \frac{|\Gamma|}{|Y|}\int_0^T\int_\Omega (g_i(u_1^1,u_2^1) - g_i(u_1^0,u_2^0)) v\, dx\, dt,\\
&|\mathrm{Err}_3(v,\varepsilon)| \leqslant \frac{|\Gamma| L_{\mathrm{g}}}{|Y|}\sum_{j=1}^{2}\|u_j^1 - u_j^0\|_{L^2(0,T;L^2(\Omega))}\|v\|_{L^2(0,T;L^2(\Omega))} \leqslant \varepsilon K_7 \sum_{j=1}^{2}\|\tilde{\mathrm{N}}^j\|_{L^\infty(Y)^d}\|\nabla u_j^0\|_{L^2(0,T;L^2(\Omega))^d}\|v\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned} \tag{55}$$

In the following, we aim at substitution of v by piecewise constant average $\langle T_\varepsilon v\rangle(x) := \langle T_\varepsilon v\rangle_{Y_j}(x)$ for $x \in \Omega_j^\varepsilon, j = 1,2$. For this task, we decompose I_i in (52) as follows:

$$I_i(v) = J_i(\langle T_\varepsilon v\rangle) + \mathrm{Err}_4(v,\varepsilon),$$

with the terms defined as

$$J_i(\langle T_\varepsilon v\rangle) := \frac{1}{|Y|}\int_0^T \int_{\Omega\times\Gamma} \left(g_i(T_\varepsilon u_1^\varepsilon, T_\varepsilon u_2^\varepsilon) - g_i(u_1^1,u_2^1)\right)\langle T_\varepsilon v\rangle\, dx\, d\sigma_y\, dt,$$

$$\begin{aligned}
\mathrm{Err}_4(v,\varepsilon) := \int_0^T \Bigg(&\int_{\Gamma^\varepsilon} \varepsilon g_i(u_1^\varepsilon,u_2^\varepsilon) v\, d\sigma_x - \frac{1}{|Y|}\int_{\Omega\times\Gamma} g_i(T_\varepsilon u_1^\varepsilon, T_\varepsilon u_2^\varepsilon)\langle T_\varepsilon v\rangle\, dx\, d\sigma_y\\
&- \frac{|\Gamma|}{|Y|}\int_\Omega g_i(u_1^1,u_2^1) v\, dx + \frac{1}{|Y|}\int_{\Omega\times\Gamma} g_1(u_1^1,u_2^1)\langle T_\varepsilon v\rangle\, dx\, d\sigma_y \Bigg) dt.
\end{aligned}$$

We apply the integration rule (20b) to the first term of Err_4 and rewrite the third term using $|\Gamma| = \int_\Gamma d\sigma_y$. Based on the boundedness (6) of g_i, from the Cauchy-Schwarz inequality, it follows the error estimate

$$
\begin{aligned}
|\mathrm{Err}_4(v,\varepsilon)| &= \frac{1}{|Y|}\left|\int_0^T\int_{\Omega\times\Gamma} g_i(T_\varepsilon u_1^\varepsilon, T_\varepsilon u_2^\varepsilon)(T_\varepsilon v - \langle T_\varepsilon v\rangle)\,dx\,d\sigma_y\,dt - \int_0^T\int_{\Omega\times\Gamma} g_i(u_1^1,u_2^1)(v-\langle T_\varepsilon v\rangle)\,dx\,d\sigma_y\,dt\right| \\
&\leqslant \|\varepsilon g_i(u_1^\varepsilon,u_2^\varepsilon)\|_{L^2(0,T;L^2(\Gamma^\varepsilon))}\frac{1}{\sqrt{|Y|}}\|T_\varepsilon v - \langle T_\varepsilon v\rangle\|_{L^2(0,T;L^2(\Omega\times\Gamma))} \\
&\quad + \frac{1}{|Y|}\|g_i(u_1^1,u_2^1)\|_{L^2(0,T;L^2(\Omega\times\Gamma))}\|v-\langle T_\varepsilon v\rangle\|_{L^2(0,T;L^2(\Omega\times\Gamma))} \\
&\leqslant \varepsilon K_8 K_{\mathrm{g}}\|\nabla v\|_{L^2(0,T;L^2(\Omega))^d},
\end{aligned}
\tag{56}
$$

where $K_8 = \sqrt{\varepsilon T|\Gamma^\varepsilon|K_{\mathrm{tr}}(1+K_{\mathrm{P}})} + \frac{|\Gamma|}{|Y|}\sqrt{T|\Omega|K_{\mathrm{P}}}$. Here, we have used the Poincaré inequality (22), following the trace inequality in periodic domains (26) such that

$$
\begin{aligned}
\int_{\Omega\times\Gamma}(T_\varepsilon v - \langle T_\varepsilon v\rangle)^2\,dx\,d\sigma_y &= \sum_{j=1}^2\int_{\Omega_j^\varepsilon\times\Gamma}(T_\varepsilon v - \langle T_\varepsilon v\rangle_{Y_j})^2\,dx\,d\sigma_y \\
&\leqslant \sum_{j=1}^2 K_{\mathrm{tr}}\int_{\Omega_j^\varepsilon\times Y_j}\left((T_\varepsilon v - \langle T_\varepsilon v\rangle_{Y_j})^2 + |\nabla_y(T_\varepsilon v)|^2\right)\,dx\,dy \\
&\leqslant K_{\mathrm{tr}}(1+K_{\mathrm{P}})\sum_{j=1}^2\int_{\Omega_j^\varepsilon\times Y_j}|\nabla_y(T_\varepsilon v)|^2\,dx\,dy \leqslant \varepsilon|Y|K_{\mathrm{tr}}(1+K_{\mathrm{P}})\int_\Omega|\nabla v|^2\,dx.
\end{aligned}
$$

Applying Young inequality to J_i implies that

$$
|J_i(\langle T_\varepsilon v\rangle)| \leqslant \frac{1}{|Y|}\int_0^T\int_{\Omega\times\Gamma}\left(\frac{1}{2}\left|g_i(T_\varepsilon u_1^\varepsilon, T_\varepsilon u_2^\varepsilon) - g_i(u_1^1,u_2^1)\right|^2 + \frac{1}{2}\langle T_\varepsilon v\rangle^2\right)\,dx\,d\sigma_y\,dt.
$$

Due to the Lipschitz continuity (7) of g_i, using the mean inequality

$$
|T_\varepsilon u_i^\varepsilon - u_i^1|^2 \leqslant 2|T_\varepsilon(u_i^\varepsilon - u_i^1)|^2 + 2|T_\varepsilon u_i^1 - u_i^1|^2,
$$

application of the integration rule (21c) and the trace inequality (25) proceeds further

$$
\begin{aligned}
|J_i(\langle T_\varepsilon v\rangle)| &\leqslant \frac{1}{|Y|}\int_0^T\int_{\Omega\times\Gamma}\left(2L_{\mathrm{g}}^2\sum_{j=1}^2|T_\varepsilon(u_j^\varepsilon - u_j^1)|^2 + \frac{1}{2}\langle T_\varepsilon v\rangle^2\right)\,dx\,d\sigma_y\,dt + \mathrm{Err}_5(v,\varepsilon) \\
&= 2\varepsilon L_{\mathrm{g}}^2\sum_{j=1}^2\int_0^T\int_{\Gamma^\varepsilon}|u_j^\varepsilon - u_j^1|^2 d\sigma_x\,dt + \frac{|\Gamma|}{2}\sum_{j=1}^2\int_0^T\int_{\Omega_j^\varepsilon}\langle T_\varepsilon v\rangle^2\,dx\,dt + \mathrm{Err}_5(v,\varepsilon) \\
&\leqslant 2K_{\mathrm{tr}}L_{\mathrm{g}}^2\sum_{j=1}^2\left(\|u_j^\varepsilon - u_j^1\|^2_{L^2(0,T;L^2(\Omega_j^\varepsilon))} + \varepsilon^2\|\nabla(u_j^\varepsilon - u_j^1)\|^2_{L^2(0,T;L^2(\Omega_j^\varepsilon))^d}\right) + \frac{|\Gamma|}{2}\sum_{j=1}^2\frac{1}{|Y_j|^2}\|v\|_{L^2(0,T;L^2(\Omega))} + \mathrm{Err}_5(v,\varepsilon),
\end{aligned}
\tag{57}
$$

because of (see Cioranescu et al[43, proposition 2.17])

$$
\|\langle T_\varepsilon v\rangle\|_{L^2(\Omega_j^\varepsilon)} = \frac{|Y|}{|Y_j|}\|T_\varepsilon^{-1}v\|_{L^2(\Omega)} \leqslant \frac{\sqrt{|Y|}}{|Y_j|}\|v\|_{L^2(\Omega)},
$$

where

$$
\mathrm{Err}_5(v,\varepsilon) := \frac{2L_{\mathrm{g}}^2}{|Y|}\sum_{j=1}^2\int_0^T\int_{\Omega\times\Gamma}\left|T_\varepsilon u_j^1 - u_j^1\right|^2\,dx\,d\sigma_y\,dt.
$$

First, we estimate Err_5 in (57). Since $u_i^1 \in H^1(\Omega)$, according to Griso,[29, formula (3.4)] the auxiliary estimate for the term in Err_5 holds:

$$\|T_\varepsilon u_j^1 - u_j^1\|^2_{L^2(\Omega\times Y_j)} \leqslant \varepsilon^2 K_c \|\nabla u_j^1\|^2_{L^2(\Omega)^d}, \quad K_c > 0.$$

Therefore, from the trace theorem (24) in $\Omega \times Y_j$ and (21b), we have

$$\frac{1}{|Y|}\|T_\varepsilon u_j^1 - u_j^1\|^2_{L^2(\Omega\times\Gamma)} \leqslant \frac{K_{\mathrm{tr}}}{|Y|}\left(\|T_\varepsilon u_j^1 - u_j^1\|^2_{L^2(\Omega\times Y_j)} + \|\nabla_y(T_\varepsilon u_j^1)\|^2_{L^2(\Omega\times Y_j)^d}\right)\varepsilon^2 K_{\mathrm{u}}\|\nabla u_j^1\|^2_{L^2(\Omega)^d}, \quad K_{\mathrm{u}} := K_{\mathrm{tr}}\left(\frac{K_{\mathrm{c}}}{|Y|}+1\right),$$

and the term $\mathrm{Err}_5(v,\varepsilon)$ is estimated by

$$0 \leqslant \mathrm{Err}_5(v,\varepsilon) \leqslant 2\varepsilon^2 L_{\mathrm{g}}^2 K_{\mathrm{u}} \sum_{j=1}^{2} \|\nabla u_j^1\|^2_{L^2(0,T;L^2(\Omega))^d}. \tag{58}$$

Let $\eta_\Omega(x)$ be a smooth cutoff function with a compact support in Ω and equals one outside an ε-neighborhood of the boundary $\partial\Omega$ such that $|\eta_\Omega| \leqslant 1$ and $\varepsilon|\nabla\eta_\Omega| \leqslant C_\eta$. For further use, we employ the following functions $w_i \in \mathcal{V}^0 \subset \mathcal{V}_i^\varepsilon$ expressed equivalently in two ways as

$$w_i := \tilde{u}_i^\varepsilon - u_i^0 - \varepsilon(T_\varepsilon^{-1}\tilde{\mathrm{N}}^i)\cdot\nabla u_i^0 \eta_\Omega = \tilde{u}_i^\varepsilon - u_i^1 + \varepsilon(T_\varepsilon^{-1}\tilde{\mathrm{N}}^i)\cdot\nabla u_i^0(1-\eta_\Omega), \tag{59}$$

where $\tilde{u}_i^\varepsilon \in H_0^1(\Omega)$ is the uniform extension of $u_i^\varepsilon \in \mathcal{U}_i^\varepsilon$ according to Lemma 3.

We will derive the estimates for $\tilde{u}_i^\varepsilon - u_i^1$ with the help of substitution of the test function $v = w_i$ from (59) into the expressions for $\mathrm{Err}_k(v,\varepsilon)$, $k = 0, 1, \ldots, 5$. This implies the following structure of the bounds:

$$|\mathrm{Err}_k(w_i,\varepsilon)| \leqslant \varepsilon\alpha_k U_k, \tag{60}$$

where the terms are defined by means of

$$\begin{aligned}
\alpha_0 &:= K\|A_i\|_{L^\infty(Y_i)}\left(\|N^i\|_{W^{1,\infty}(Y_i)^d}+1\right), & U_0 &:= \|u_i^0\|_{L^2(0,T;H^3(\Omega_i^\varepsilon))}\|w_i\|_{L^2(0,T;H^1(\Omega_i^\varepsilon))},\\
\alpha_1 &:= \|N^i\|_{L^\infty(Y_i)^d}, & U_1 &:= \|\partial_t(\nabla u_i^0)\|_{L^2(0,T;H^1(\Omega_i^\varepsilon)^*)^d}\|w_i\|_{L^2(0,T;H^1(\Omega_i^\varepsilon))},\\
\alpha_2 &:= K_6K_{\mathrm{g}}, & U_2 &:= \|w_i\|_{L^2(0,T;H^1(\Omega))},\\
\alpha_3 &:= K_7\textstyle\sum_{j=1}^2\|\tilde{\mathrm{N}}^j\|_{L^\infty(Y)^d}, & U_3 &:= \textstyle\sum_{j=1}^2\|\nabla u_j^0\|_{L^2(0,T;L^2(\Omega))^d}\|w_j\|_{L^2(0,T;L^2(\Omega))},\\
\alpha_4 &:= K_8K_{\mathrm{g}}, & U_4 &:= \|\nabla w_i\|_{L^2(0,T;L^2(\Omega))^d},\\
\alpha_5 &:= 2\varepsilon L_{\mathrm{g}}^2K_{\mathrm{u}}, & U_5 &:= \textstyle\sum_{j=1}^2\|\nabla u_j^1\|^2_{L^2(0,T;L^2(\Omega))^d},
\end{aligned}$$

According to the uniform estimate (9) in Theorem 1 and the continuous extension (27), we have

$$\|w_i\|^2_{L^2(0,T;H^1(\Omega))} \leqslant 3K_{\mathrm{e}}\|u_i^\varepsilon\|^2_{L^2(0,T;H^1(\Omega_i^\varepsilon))} + 3\|u_i^0\|^2_{L^2(0,T;H^1(\Omega))} + 3\varepsilon\|\tilde{\mathrm{N}}_i\|_{L^\infty(Y)^d}\|\sqrt{\varepsilon}\nabla u_i^0\eta_\Omega\|^2_{L^2(0,T;H^1(\Omega))^d} = O(1) \tag{61}$$

following that all $\alpha_k = O(1)$ and $U_k = O(1)$ for $k = 0, 1, \ldots, 5$.

The asymptotic equation (51) tested with the function $v = w_i$ from (59) leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^T\int_{\Omega_i^\varepsilon}(u_i^\varepsilon - u_i^1)^2\,dx\,dt + \int_0^T\int_{\Omega_i^\varepsilon}A_i^\varepsilon\nabla(u_i^\varepsilon - u_i^1)\cdot\nabla(u_i^\varepsilon - u_i^1)\,dx\,dt\\
&\quad = J_i(\langle T_\varepsilon w_i\rangle) + \sum_{k=0}^{4}\mathrm{Err}_k(w_i,\varepsilon) + \mathrm{Err}_6(\varepsilon) + M(u_i^\varepsilon - u_i^1)
\end{aligned} \tag{62}$$

with the following two terms:

$$\mathrm{Err}_6(\varepsilon) := -\int_0^T \langle \partial_t(u_i^\varepsilon - u_i^1), \varepsilon(T_\varepsilon^{-1}N_i)\cdot\nabla u_i^0(1-\eta_\Omega)\rangle_{\Omega_i^\varepsilon}\,dt,$$

$$M(u_i^\varepsilon - u_i^1) := -\int_0^T\int_{\Omega_i^\varepsilon} A_i^\varepsilon \nabla(u_i^\varepsilon - u_i^1)\cdot\nabla[\varepsilon(T_\varepsilon^{-1}N^i)\cdot\nabla u_i^0(1-\eta_\Omega)]\,dx\,dt.$$

We note that M is not an error term; in contrary, it enters with the factor $-\delta_1$ the left-hand side of the estimate (65) following later.

Err_6 is estimated by integration by parts with respect to time

$$\mathrm{Err}_6(\varepsilon) = \int_0^T\int_{\Omega_i^\varepsilon}(u_i^\varepsilon - u_i^1)\varepsilon(T_\varepsilon^{-1}N_i)\cdot\partial_t(\nabla u_i^0)(1-\eta_\Omega)\,dx\,dt - \int_{\Omega_i^\varepsilon}(u_i^\varepsilon - u_i^1)\varepsilon(T_\varepsilon^{-1}N_i)\cdot\nabla u_i^0(1-\eta_\Omega)\,dx\Big|_{t=0}^T,$$

after using Young inequality and the continuous embedding

$$\|u_i^\varepsilon - u_i^1\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))} \leqslant K_{\mathrm{emb}}\|u_i^\varepsilon - u_i^1\|^2_{L^\infty(0,T;L^2(\Omega_i^\varepsilon))}, \tag{63}$$

which implies that

$$|\mathrm{Err}_6(\varepsilon)| \leqslant \varepsilon\alpha_6 U_6,$$

where

$$\alpha_6 := \frac{2+K_{\mathrm{emb}}}{2}\|N^i\|_{L^\infty(Y_i)^d},$$
$$U_6 := \frac{1}{2+K_{\mathrm{emb}}}\left(\|\partial_t(\nabla u_i^0)(1-\eta_\Omega)\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))^d} + 2\|\nabla u_i^0(1-\eta_\Omega)\|^2_{L^\infty(0,T;L^2(\Omega_i^\varepsilon))^d}\right) + \|u_i^\varepsilon - u_i^1\|^2_{L^\infty(0,T;L^2(\Omega_i^\varepsilon))}.$$

The term $M(u_i^\varepsilon - u_i^1)$ is evaluated by Young inequality with the weight $\delta_1 > 0$ and using the boundedness property of A_i with the upper bound β from (5) as

$$|M(u_i^\varepsilon - u_i^1)| = \Bigg|\int_0^T\int_{\Omega_i^\varepsilon} A_i^\varepsilon\nabla(u_i^\varepsilon - u_i^1)\cdot\{T_\varepsilon^{-1}(\partial_y N^i)\cdot\nabla u_i^0(1-\eta_\Omega) + \varepsilon(T_\varepsilon^{-1}N^i)\cdot\partial_x(\nabla u_i^0)(1-\eta_\Omega)$$
$$-\varepsilon(T_\varepsilon^{-1}N^i)\cdot\nabla u_i^0\nabla\eta_\Omega\}\,dx\,dt\Bigg| \leqslant \frac{3\beta\delta_1}{2\sqrt{3}}\|\nabla(u_i^\varepsilon - u_i^1)\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))^d} + \mathrm{Err}_7(\varepsilon),$$

where

$$\mathrm{Err}_7(\varepsilon) := \frac{\sqrt{3}\beta}{2\delta_1}\Big\{\|\partial_y N^i\|_{L^\infty(Y_i)^{d\times d}}\|\nabla u_i^0(1-\eta_\Omega)\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))^d}$$
$$+\varepsilon^2\|N^i\|_{L^\infty(Y_i)^d}\|\partial_x(\nabla u_i^0)(1-\eta_\Omega)\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))^{d\times d}} + \varepsilon^2\|N^i\|_{L^\infty(Y_i)^d}\|\nabla u_i^0\cdot\nabla\eta_\Omega\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))}\Big\}.$$

It follows

$$|\mathrm{Err}_7(\varepsilon)| \leqslant \varepsilon\alpha_7 U_7,$$

where

$$\alpha_7 := \frac{\sqrt{3}\beta}{2}\left(\|N^i\|_{L^\infty(Y_i)^d} + \|\partial_y N^i\|_{L^\infty(Y_i)^{d\times d}}\right),$$
$$U_7 := \frac{1}{\delta_1}\left(\|\nabla u_i^0(1-\eta_\Omega)\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))^d} + \varepsilon\|\partial_x(\nabla u_i^0)(1-\eta_\Omega)\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))^{d\times d}} + \varepsilon\|\nabla u_i^0\cdot\nabla\eta_\Omega\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))}\right) = O(1).$$

We note that $U_7 = O(1)$, in particular, because $1-\eta_\Omega \neq 0$ on a $O(\varepsilon)$-set using the fact that $1-\eta_\Omega \neq 0$ on a set of measure $O(\varepsilon)$, thus compensating $\nabla\eta_\Omega = O(\varepsilon^{-1})$ here.

Therefore, using the inequality (57) for $J_i(\langle T_\varepsilon w_i\rangle)$ and the uniform positive definiteness (33) of A_i with the lower bound $\alpha > 0$, from (62), we arrive at the estimate

$$
\begin{aligned}
&\left| \frac{1}{2}\int_{\Omega_i^\varepsilon} (u_i^\varepsilon - u_i^1)^2 \Big|_{t=0}^{T} dx + \left(\alpha - \frac{\sqrt{3}\beta\delta_1}{2}\right) \int_0^T \int_{\Omega_i^\varepsilon} |\nabla(u_i^\varepsilon - u_i^1)|^2 \,dx\,dt \right| \\
&\leqslant (2K_{\mathrm{tr}}L_{\mathrm{g}}^2 + \alpha_8)\sum_{i=1}^{2} \|u_i^\varepsilon - u_i^1\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))} + 2\varepsilon^2 K_{\mathrm{tr}}L_{\mathrm{g}}^2 \sum_{i=1}^{2} \|\nabla(u_i^\varepsilon - u_i^1)\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))^d} \\
&+ \sum_{k=0}^{5} |\mathrm{Err}_k(w_i,\varepsilon)| + \sum_{k=6}^{8} |\mathrm{Err}_k(\varepsilon)|,
\end{aligned} \tag{64}
$$

where $\alpha_8 := \frac{|\Gamma|}{2}\sum_{j=1}^{2} \frac{1}{|Y_j|^2}$, and

$$
0 \leqslant \mathrm{Err}_8(\varepsilon) := \alpha_8 \|\varepsilon(T_\varepsilon^{-1}N^i)\cdot \nabla u_i^0 (1-\eta_\Omega)\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))} \leqslant \varepsilon^2 \alpha_8 \|N^i\|_{L^\infty(Y_i)^d} \|\nabla u_i^0\|_{L^2(0,T;L^2(\Omega_i^\varepsilon))^d}.
$$

After summation over $i = 1, 2$ we rearrange the terms such that

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{2} \|(u_i^\varepsilon - u_i^1)(T)\|^2_{L^2(\Omega_i^\varepsilon)} + \gamma \sum_{i=1}^{2} \|\nabla(u_i^\varepsilon - u_i^1)\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))^d} \leqslant \alpha_{10} \sum_{i=1}^{2} \|u_i^\varepsilon - u_i^1\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))} + \mathrm{Err}_{10}(\varepsilon), \\
&\mathrm{Err}_{10}(\varepsilon) := \sum_{k=0}^{5}\sum_{i=1}^{2} |\mathrm{Err}_k(w_i,\varepsilon)| + 2\sum_{k=6}^{9} |\mathrm{Err}_k(\varepsilon)|,
\end{aligned} \tag{65}
$$

where $\gamma := \alpha - 4\varepsilon^2 K_{\mathrm{tr}} L_{\mathrm{g}}^2 - \frac{\sqrt{3}\beta\delta_1}{2}$, $\alpha_{10} := 2(K_{\mathrm{tr}}L_{\mathrm{g}}^2 + \alpha_8)$, and the error Err_9 implies

$$
\mathrm{Err}_9(\varepsilon) := \frac{1}{2}\|(u_i^\varepsilon - u_i^1)(0)\|^2_{L^2(\Omega_i^\varepsilon)} \leqslant \frac{\varepsilon}{2} \|N^i\|_{L^\infty(Y_i)^d} \|\nabla u_i^0(0)\|^2_{L^2(\Omega_i^\varepsilon)^d} = O(\varepsilon).
$$

After taking the supremum over time, using the embedding theorem (63), we estimate the first term in the left-hand side of (65) by the lower bound

$$
\frac{1}{2}\sum_{i=1}^{2} \|(u_i^\varepsilon - u_i^1)(T)\|^2_{L^2(\Omega_i^\varepsilon)} \geqslant \frac{1}{4K_{\mathrm{emb}}} \sum_{i=1}^{2} \|u_i^\varepsilon - u_i^1\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))} + \frac{1}{4}\sum_{i=1}^{2} \|u_i^\varepsilon - u_i^1\|^2_{L^\infty(0,T;L^2(\Omega_i^\varepsilon))}.
$$

We continue the estimate (65) by taking δ_1 small enough such that $\gamma > 0$. Therefore, applying Grönwall inequality leads to

$$
\sum_{i=1}^{2} \|(u_i^\varepsilon - u_i^1)(t)\|^2_{L^2(\Omega_i^\varepsilon)} \leqslant \mathrm{Err}_{11}(\varepsilon), \quad \mathrm{Err}_{11}(\varepsilon) := 2\mathrm{Err}_{10}(\varepsilon)\exp(2\alpha_{10}T).
$$

As a consequence, from (65) and the embedding theorem (63), we conclude with the estimate

$$
\begin{aligned}
&\sum_{i=1}^{2} \|u_i^\varepsilon - u_i^1\|^2_{L^\infty(0,T;L^2(\Omega_i^\varepsilon))} + \sum_{i=1}^{2} \|\nabla(u_i^\varepsilon - u_i^1)\|^2_{L^2(0,T;L^2(\Omega_i^\varepsilon))^2} \leqslant \mathrm{Err}_{12}(\varepsilon), \\
&\mathrm{Err}_{12} := \frac{2}{\min\left(\frac{1}{2}, \frac{1}{2K_{\mathrm{emb}}}, \gamma\right)} (\alpha_{10}\mathrm{Err}_{11}(\varepsilon) + \mathrm{Err}_{10}(\varepsilon)) = O(\varepsilon),
\end{aligned} \tag{66}
$$

which finishes the proof. □

7 | DISCUSSION

Compared with previous results in the literature on multiscale diffusion equations, in the paper, we derived the macroscopic bidomain model that is advantageous for numerical simulation; we first proved the homogenization result supported by residual error estimate of the asymptotic corrector due to the nonlinear transmission condition at the microscopic level, which appears to describe interface chemical reactions.

For further generalization of the obtained result, we suggest to consider the case of connected-disconnected domains Ω_1^ϵ and Ω_2^ϵ. While in the connected domain Ω_1^ϵ the uniform extension is applicable, the disconnected domain Ω_2^ϵ allows a discontinuous Poincaré estimate (see Kovtunenko and Zubkova[21]).

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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